Journal of Applied Mathematics and Mechanics 72 (2008) 296–302

Contents lists available at ScienceDirect

Journal of Applied Mathematics and Mechanics

journal homepage: www.elsevier.com/locate/jappmathmech

Bifurcation of multimode flows of a viscous fluid in a plane diverging channel☆

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ARTICLE INFO

Article history:
Received 14 April 2007

ABSTRACT

The pattern of steady multimode flow of a viscous incompressible fluid in a plane diverging channel is constructed and investigated. It is shown that odd-mode flows have velocity profiles that are symmetrical about the axis of the channel and from one to three different flows with a fixed number of modes exist. The even-mode flows are asymmetric and exist as pairs. The existence of a denumerable set of finite ranges adjoining one another, in which a single-type of complex bifurcation of the flow occurs, is established in the case of an unbounded range of values of the Reynolds number. As the Reynolds number increases, transitions to flows with an increasing number of modes, containing domains of forward and backward flows, occur successively. Flow patterns with a smaller number of modes do not occur. An increase in the number of an range corresponding to an increase in the Reynolds number leads to an unlimited increase in the length of the range and the number of modes of permissible flows.

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1. Formulation of the problem and introductory remarks

The steady flow of a viscous incompressible fluid in a plane diverging channel[1–4] is investigated, that is, in the domain

$$\Omega = \{(r,\theta):\, r>0,\, |\theta|<\beta\}, \quad 0<\beta\le\pi \tag{1.1}$$

where r and θ are polar coordinates and β is the semi-aperture angle. No-slip conditions are satisfied on the channel walls $r>0$, $\theta=\pm\beta$ and, when $r=0$, the flow has a source-type singularity. In the classical formulation of the problem, due to Jeffery[1] the power of the source (we subsequently assume that $Q>0$) is assumed to be specified, that is, the flow rate across any section $r=\text{conts}>0$ of the domain Ω (1.1) is fixed.

Note that this formulation of the problem (see below) and its solution are traditionally also associated with the investigations of Hamel[5] or (more frequently) are referred to as the Jeffery–Hamel problem and Jeffery–Hamel flows.[2–4,6–15] However, an analysis of the results in Ref. 5 suggests that they do not contain a rigorous formulation of the boundary value problem with the condition of constancy of the fluid flow rate and boundary conditions. There is no concept of a Reynolds number in it and the possibility of radial and spiral plane flows is analysed. An implicit analytical solution of the problem of plane radial, circular and spiral flows was obtained by Jeffery.[1]

The system only permits the introduction of two dimensionless parameters: the aperture angle of the diverging channel 2β and the Reynolds number $\mathrm{Re}=\rho Q/\mu$, where ρ is the density and μ is the dynamic viscosity of the fluid. This property does not enable the equations of motion to be reduced to a completely dimensionless form[1–4] but enables a self-similar solution to be constructed with a radial velocity field of the form

$$\upsilon_r = \frac{Q}{r}V(\theta), \quad \upsilon_\theta \equiv 0, \quad (r,\theta)\in\Omega \tag{1.2}$$

This solution satisfies the incompressibility condition in the case of a dimensionless, as yet unknown, smooth function $V(\theta)$.

The relations,[1–4] defining the velocity profile $V(\theta)$,

$$V'' + 4V + \mathrm{Re}V^2 = C = \text{const}, \quad |\theta|<\beta, \quad V(\pm\beta) = 0 \tag{1.3}$$

☆ *Prikl. Mat. Mekh.* Vol. 72, No. 3, pp. 431–441, 2008.

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doi:10.1016/j.jappmathmech.2008.07.007

$$\int_{-\beta}^{\beta} V(\theta)d\theta = 1 \tag{1.4}$$

follow from the Navier–Stokes equations, the no-slip conditions on the walls of the diverging channel (1.1) and the constancy of the fluid flow rate.

Boundary-value problem (1.3) and the integral condition (1.4) serve to find the integration constants and the unknown C for specified values of the parameter β and Re. The pressure p, corresponding to a solution of the form of (1.2), is expressed in terms of the quantities V and C which have been found:

$$p = -\frac{\rho Q^2}{2r^2\mathrm{Re}}(C - 4V), \quad (r,\theta) \in \Omega \tag{1.5}$$

According to expression (1.5), the constant C defines the pressure $p(\pm\beta)$ on the walls of the diverging channel.

The non-linear boundary value problem (1.3) with the additional integral condition (1.4) is symmetrical under the transformation $\theta \to -\theta \to -\beta$. However, it has been explicitly established by the authors (this property was noted earlier, see Refs. 2, 3, etc) that, in the case of flows in a converging channel $Q < 0$[16,17] and a diverging channel $Q > 0$[18,19], this problem, together with symmetric odd-mode solutions, has even-mode solutions which are asymmetric about the axis. This property is a consequence of the axial symmetry of the problem and it testifies to the channel walls (boundaries) "having equal rights" We note that a recent, detailed and constructive analysis of the effect of symmetry properties in the Navier–Stokes equations is available.[4]

The pattern of bifurcations of single-mode, purely radial flows for a diverging channel has been constructed by the authors[19] for comparatively small values of $\mathrm{Re}\alpha(2\beta\mathrm{Re} \le 6\pi)$ and a wide range of variation in the semi-aperture angle $\beta(0 < \beta \le \pi/2)$. The announcement[18] of investigations for a small semi-aperture angle $0 < \beta \le 10^\circ$ suggests that, when Re is increased, only multimode flows, containing domains of forward and backward flows, are possible. It has been established that the number of modes of possible flows increases in a specific manner (see later) and has a lower limit. A clear picture of the bifurcation of the multimode flows of an incompressible viscous fluid in a plane diverging channel, which does not exist in the literature, is presented below on the basis of a complete, rigorous solution of the Jeffery problem.

We shall get rid of the integral relation in problem (1.3), (1.4) by introducing of an unknown Z which characterizes the flow rate of the fluid in the angle $\theta \ge -\beta$. By differentiation of the equation for V, boundary value problem (1.3) with integral condition (1.4) is reduced to the equivalent fourth-order boundary value problem

$$V'''' + 4V' + 2\mathrm{Re}VV' = 0, \quad Z' = V - \frac{1}{2\beta}; \quad V(\pm\beta) = Z(\pm\beta) = 0 \tag{1.6}$$

It is required to construct the solution V, V', C of problem (1.3), (1.4) or the solution V, V', V'', Z of the equivalent problem (1.6) and, also, to find the velocity $\upsilon(r,\theta)$ (1.2), the pressure p (1.5) and the other kinematic and dynamic characteristics of the flows as a function of r and θ and the governing parameters β and Re for the whole domain Ω (1.1) when $\mathrm{Re} > 0$, $0 < \beta \le \pi$. Note that, in the literature (see Refs. 1–3, etc), it is usually assumed that $Q > 0$ for a flow in a diverging channel (source-type singularity). In the case of a converging channel (a sink-type singularity), it is necessary to change the plus sign to a minus sign in expressions (1.2), (1.3) and (1.6) in front of the coefficients of Q and Re respectively and, also, to change the sign of the magnitude of the pressure p (1.5).

2. Numerical and analytical solution of the Jeffery boundary value problem

The formal analytical solution of problem (1.3), (1.4) using a first integral leads to elliptic functions and integrals, which depend on three unknown parameters[1–3,8]. A system of two transcendental equations that also contain two independent parameters (β and Re), which can be specified in the solution, can be obtained for determining the unknown integration constants. Note that not one of the above-mentioned parameters is expressed explicitly in terms of the remaining parameters, which does not permit one to reduce the dimension of the system and to analyse it. This solution is essentially implicit and its use leads to fundamental computational difficulties resulting from the degeneration of the equations when $\mathrm{Re} \to 0$, $\mathrm{Re} \to \infty$ and, also, from the bifurcations of the solutions for a denumerable set of positive values of the number Re. Furthermore, the dependence of the solution on the angle β is also non-regular.[16–19]

These computational difficulties are aggravated by the insufficient accuracy (10^{-4}–10^{-6}, whereas an accuracy of 10^{-8}–10^{-10} and higher is required) of the tabulated data for elliptic functions and integrals. Hence, the procedure of reduction to a system of transcendental equations is unnecessary since it hinders a constructive solution of the boundary value problem.

The incompleteness of its investigation is explained by the lack of adequate, high-accuracy methods for calculating the solutions of a problem containing internal and boundary singularities. An analogous situation arises when investigating the solution in the case of flows in a converging channel ($Q < 0$) which has been previously studied in detail in Refs, 16 and 17. The numerical results known in the literature involve the solution of boundary-value problem (1.3), (1.4), ignoring the integral condition (1.4). In this approach, the fluid velocity on the axis of the diverging or coverging channel is usually specified (see Refs. 6, 7, etc), which artificially constricts the class of solutions.

An approach has been developed for the direct solutions of non-linear boundary-value problems of the type (1.3), (1.4) and (1.6) on the basis of which a constructive numerical-analytical method for calculating the unknown constants is proposed.[20] It contains an accelerated convergence algorithm and a procedure for continuation with respect to the parameters. The normalized unknown functions y and z, the argument x and the unknown parameters γ and λ, i.e.,

$$y(x) = 2\beta V(\theta), \quad z = 2\beta Z, \quad x = \frac{1}{2}\left(\frac{\theta}{\beta} + 1\right), \quad 0 \le x \le 1; \quad \gamma = y'(0), \quad \lambda = 8\beta^3 C \tag{2.1}$$

are introduced in relations (1.3) and (1.4) to standardize the calculations and to make them more convinient. These quantities are uniquely related to the initial V, Z, Q, $V'(-\beta)$, C. The unknown functions y, y', z and the parameters γ and λ are determined by solving the non-linear boundary value problem for fixed values of the parameters $a = 4\beta$ and $b = -2\beta\mathrm{Re}$:

$$y'' + a^2 y - by^2 = \lambda, \quad y(0) = 0, \quad y'(0) = \gamma$$
$$z' = y - 1, \quad z(0) = 0; \quad y(1) = z(1) = 0 \tag{2.2}$$

Problem (2.2) contains four unknown parameters: the three constants of integration of the differential equations (the value of γ is taken as one of them) and the parameter λ. Four null boundary conditions when $x = 0, 1$ are available for finding them.

Suppose an approximate solution (which is not necessarily the exact solution) of boundary-value problem (2.2) is known for the specified values $a = a_0$, $b = b_0$. We then apply the method of accelerated convergence algorithm[20] for $a = a_0$ and the new value $b = b_0 + \delta b$, where δb is a sufficiently small quantity. It consists of refining the unknowns γ and λ using a recursion scheme.

Calculations confirm the rapid convergence and efficiency of the algorithm which enables one, without the use of transcendental expressions, to carry out high accuracy operational mass calculations (see later). They enable a complete investigation of flows to be carried out, including the determination of their velocity profile[16,17]. The basic result involves finding the values of the parameters γ and λ with a geometrical and physical meaning which are required to integrate Cauchy problem (2.2). The pattern of bifurcations, depending on the parameters β and Re, is of intrinsic interest. There are no such results in the literature.

The relations $\gamma(b)$, $\lambda(b)$ were constructed as the result of calculations for different values of the angle β (that is, of the parameter a) and a set of values of the parameter $b = -2\beta\mathrm{Re}$. They enable one to determine the velocity profile according to the first formula of (2.1) as the solution of Cauchy problem (2.2).

All the kinematic and dynamic characteristics of the flows are calculated on this basis and a complete qualitative investigation of the flows in the diverging channel is carried out. The results of the calculations of the velocity profiles and the distribution of the pressure forces on the walls were obtained earlier for a converging channel[17,20]. In particular, the pressure p is determined from the formulae (1.4) and (2.1):

$$p = p(r, \theta, \beta, Q, \rho, \mu) = 2\frac{\rho Q^2}{r^2}\frac{\lambda - a^2 y}{a^2 b} \tag{2.3}$$

The quantities $y(x, a, b)$ and $\gamma(b)$, $\lambda(b)$ are assumed to be calculated using relations (2.1) and (2.2) and the previously described Scheme.[18–20] It follows from an analysis of the last dimensionless factor of expression (2.3) that the magnitude of the pressure p increases without limit when the parameter $|b|$ is reduced.

3. Analysis of flows in a diverging channel

The complete pattern of the dependence of the parameters γ and λ on the specified values of $\beta \in (0, \pi)$ and $b \le 0$ is extremely diverse and its detailed description is difficult. As in the case of flows in a converging channel,[17] it is convenient to subdivide the range of values of the semi-aperture angle of the diverging channel β into several ranges corresponding the different qualitative behaviour of the variables $\gamma(b)$ and $\lambda(b)$ in the procedure of continuation with respect to the parameter $b < 0$, $|b| \ll 1$:

$$1)\ 0 < \beta \le \frac{\pi}{4}, \quad 2)\ \frac{\pi}{4} < \beta \le \frac{\pi}{2}, \quad 3)\ \frac{\pi}{2} < \beta < \beta^*, \quad 4)\ \beta^* < \beta \le \frac{3\pi}{4}, \quad 5)\ \frac{3\pi}{4} < \beta \le \pi \tag{3.1}$$

In the case of the critical value β^*, where $\beta^* \approx 2.2467$ is the positive root of the equation $\mathrm{tg}2\beta = 2\beta$, a solution does not exist for $b = 0$ ($\mathrm{Re} = 0$) and a continuation of the solution with respect to the parameter β, $\beta \lesseqgtr \beta^*$ is possible for $b \neq 0$. The limiting values $\beta \to +0$, $\beta^* \pm 0$ and $\beta = \pi/2$, π (3.1) are also interesting from a hydrodynamic point of view.

Together with the solutions which are regularly continued with respect to b, that is, with respect to Re in the domain of values of β (3.1), We established the existence of a set of multimode flows in a converging channel[17] and diverging channel[19] which are not continued when $\mathrm{Re} \to 0$.

It is convenient to illustrate the pattern of the bifurcations of steady flows in a diverging channel when $\mathrm{Re} \to \infty$, which is unknown in the literature, in the case of slightly diverging walls (the angle β in the range from 5° to 30°) which is of interest in its own right.

The results of calculations using the technique proposed earlier,[18,20] when $\beta = 10° \approx 0.1745$ are presented later. For values of $\beta \in (0, 10°)$, the behaviour of the curves $\gamma(b)$ and $\lambda(b)$ turns out to be extremely close to the case when $\beta = 10°$. The qualitative similarity holds up to values of $\beta < 90°$.[19] Note that, for $\beta = 90°$, the flow domain is a half-plane with an aperture. When $b \to -\infty$, that is, when $\mathrm{Re} \to -\infty$, sequences of triple bifurcation points $b_*^{(n)}$ and turning points $b^{(n)*}$ $(n = 1, 2, \ldots)$ of the curves $\gamma_k^{(n)}(b)$ and $\lambda_k^{(n)}(b)$ arise, and sequences of the corresponding values of b, which will be described below. The superscript (n) indicates the number of the range, and the subscript k indicates the corresponding number of modes, that is, the number of domains of the forward and backward flows.

The flow pattern is quite complicated[18] and requires a careful step-by-step presentation. In essence (and traditionally), it is assumed to be natural to begin it from the low values of $n = 1, 2$ which corresponds to the basic[19] and following[18] ranges of values of β and Re. The bifurcation pattern is shown in Fig. 1, according to which there is a set of curves $\gamma_k^{(n)}(b)$ (the solid curves) and $\lambda_k^{(n)}(b)$ (the dashed curves), $n = 1, 2, \ldots$. Analysis of the solutions of the non-linear boundary-value problem suggests that the subscript k can take three values: $2n \mp 1$ and $2n$ for each value of n. In order to describe the flow processes it is advisable to introduce ranges of values of the parameter $b \le 0$. Within these ranges, there is substantial qualitative reorganization (bifurcation of the velocity) of the flows when the parameter b is reduced, that is, when Re is increased.

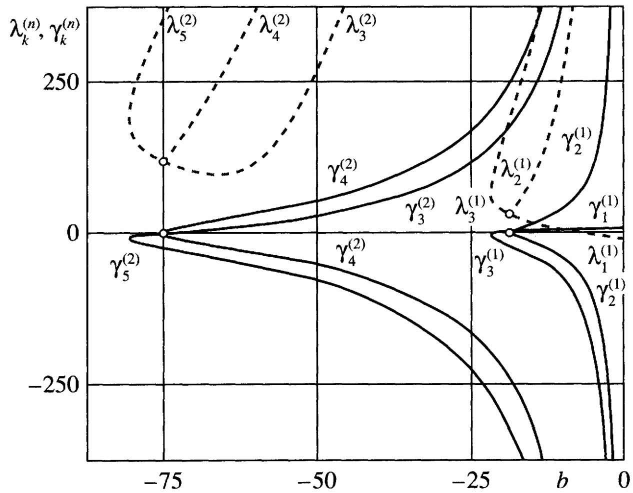

Fig. 1.

The first region ($n=1$) of values of (b, γ), (b, λ) has been investigated in detail[19] using of the procedure of continuation with respect to the parameter $b \in [b^{(1)^*}, 0]$, ($b^{(1)^*} \approx -21.7$) and the known analytical solution for $b=0$, including for larger values of β. The existence of single-mode $\gamma_1^{(1)}(b)$, $\lambda_1^{(1)}(b)$ and triple-mode $\gamma_3^{(1)}(b)$, $\lambda_3^{(1)}(b)$ flows has been established for which purely radially divergent ($y>0$) flows or flows, which are divergent ($y>0$) in the central part and convergent ($y<0$) along the boundaries of the diverging channel, occur respectively. In the graphs shown below, each curve $y_k^{(n)}(x)$ is actually the fluid velocity profile on an arc of a circle of arbitrary radius with centre at the vertex of the diverging channel. The order k of a flow mode is determined by the number of intersections of the abscissa by the function (the number of internal nodes is equal to $k-1$). At the same time, the integral of any function $y_k^{(n)}(x)$ in the interval $0 \le x \le 1$ is equal to the normalized power of the source $Q=1$. As has been pointed outs the subscript denotes the number of the range ($n=1$, 2) of the change in the parameter $b=-2\beta\mathrm{Re}$.

A single-mode flow exists when $-18.8 \approx b_*^{(1)} \le b \le 0$ and the corresponding triple-mode flow when $b_1^{(1)*} \le b < 0$. In the range $b_1^{(1)*} < b < b_*^{(1)}$, the functions $\gamma_3^{(1)}$ and $\lambda_3^{(1)}$ are two-valued, which suggests the previously unknown possibility of two different triple-mode flows being generated by a single-mode flow. These flows disappear when b, $b < b^{(1)^*}$ becomes smaller. A single triple-mode flow, corresponding to $\gamma_3^{(1)}$ and $\lambda_3^{(1)}$, exists for $b \in [b_*^{(1)}, \ 0]$ and, also, as has been noted, one single-mode flow. Moreover, two previously unknown two-mode asymmetric flows (the possibility of which was postulated in Refs. 2 and 9) exist in the above-mentioned range: there is a forward flow of the fluid close to one of the walls and a backward flow close to the other wall (and vice versa). These flows occur $b_*^{(1)} < b < 0$ and subsequently, when the parameter $b<0$ decreases, they degenerate when $b = b_*^{(1)}$ and disappear for $b < b_*^{(1)}$.

Note that the quantities $|\gamma_2^{(1)}|$ and $|\lambda_2^{(1)}|$, and, also, $|\gamma_3^{(1)}|$ and $|\lambda_3^{(1)}|$ approach infinity extremely rapidly when $b \to -0$, that is, the corresponding curves (and flows) cannot be obtained using the procedure of continuation with respect to the parameter b for $b \le 0$. In particular, when $b=-1$, we have $\gamma_3^{(1)} \sim -10^3$, $\lambda_3^{(1)} \sim 10^4$. Hence, the first range is characterized by two critical values of b. The value $b_*^{(1)}$ is a triple point at which bifurcation of one-, two- and three-mode flows occurs. Moreover, $\gamma_1^{(1)} = 0$ at this point. The quantity $b^{(1)^*}$ corresponds to the turning point of the curves $\gamma_3^{(1)}(b)$ and $\lambda_3^{(1)}(b)$ and determines the boundary of the first range. The series of curves $\gamma_k^{(1)}$, $\lambda_k^{(1)}$ ($k=1, 2, 3$) corresponding to the first bifurcation point has for the most part been described in Ref. 19. The basic qualitative property of the flow pattern when $b_*^{(1)} < b < 0$ consists of the existence of a stable single-mode flow $y_1^{(1)}(x)$.

The second bifurcation point $b_*^{(2)}$ can be successfully determined using a numerical-analytical procedure,[20] and the second range of values of b:$b^{(2)^*} \le b \le b^{(1)^*}$ found on the basis of it where, as it turns out, three-, four- and five-mode flows are possible; one- and two-mode processes do not exist. The qualitative behaviour of the curves $\gamma_k^{(2)}(b)$ and $\lambda_k^{(2)}(b)$ is in many respects similar to the behaviour of the curves $\gamma_k^{(1)}(b)$ and $\lambda_k^{(1)}(b)$ for the corresponding values of the subscript k (see Fig. 1). However, unlike the functions $\gamma_1^{(1)}$ and $\lambda_1^{(1)}$, the limiting values of which are bounded when $b=-0$, the functions corresponding to them $\gamma_3^{(2)} \to -\infty \lambda_3^{(2)} \to \infty$ when $b \to -0$. Furthermore, the curves $|\gamma_k^{(2)}|$ and $|\lambda_k^{(2)}|$ increase and tend to a vertical asymptote considerably more rapidly when $b \to -0$. For example, the values $\gamma_3^{(1)} \approx -70$, $\lambda_3^{(1)} \approx 10^3$ and the values $\gamma_5^{(2)} \approx -670$, $\lambda_5^{(2)} \approx 10^4$ corresponding to them follow from the calculations when $b=-10$ (see Fig. 1).

The second range ($n=2$) has two similar critical points which prove to be the end points of the second range $b^{(2)*}$, $b_*^{(2)}$. The value $b=b^{(2)*}$ is a turning point of the curves $\gamma_5^{(2)}$, $\lambda_5^{(2)}(b^{(2)*} \approx -80.44)$. The value $b_*^{(2)}$ is a (triple) bifurcation point, in the neighbourhood of which the indicated three branches of the steady-state flows ($b_*^{(2)} \approx -75.39$) exist. Moreover, $\gamma_3^{(2)} = 0$ at this point. In the range $b^{(2)*} < b < b_*^{(2)}$, the functions $\gamma_5^{(2)}$ and $\lambda_5^{(2)}$ are two-valued, and two five-mode flows correspond to each value of b. The functions are single-valued when $b_*^{(2)} \le b < 0$, that is, a single flow pattern of the above-mentioned form occurs. There is an analogy with the two three-mode flows pattern for the domain with index $n=1$. The four-mode flows (two of them when $b > b_*^{(2)}$ contain alternating ranges of forward and backward flows and a forward flow close to one wall induces a backward flow close to the other wall. On the axis of the diverging channel $y_4^{(2)} = 0$. Half of the flow rate is in each half of the diverging channel. This effect is significant from a hydromechanical point of view since, as was noted above, it also proves that, when the velocity on the axis of a diverging channel or a converging is specified[6,7], the class of solutions with zero velocity on the channel axis is discarded.

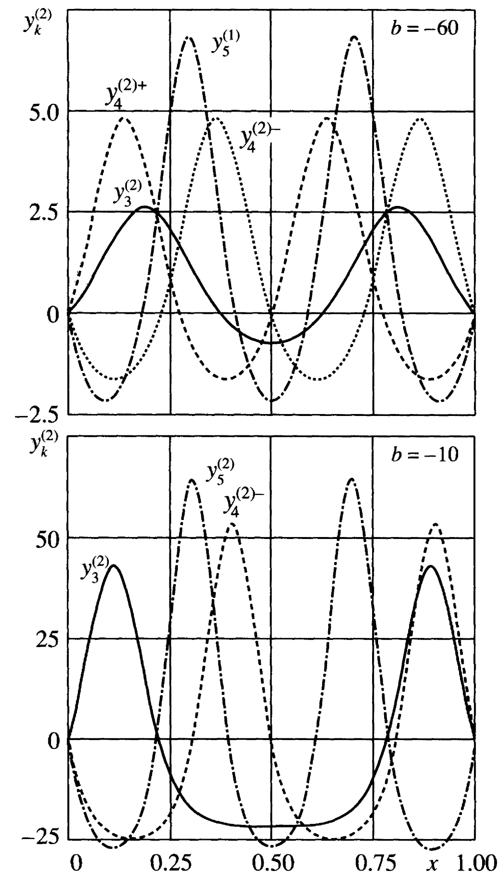

Fig. 2.

It follows from Fig. 1 that a single three-mode flow, corresponding to the curves $\gamma_3^{(2)}$ and $\lambda_3^{(2)}$, exists in the range $b_*^{(2)} < b < b^{(1)*}$. It is extendable into the range $b^{(1)*} \le b < 0$. When $b = b^{(1)*}$, there are two flows ($\gamma_3^{(1)}$ and $\lambda_3^{(1)}$ are added), in the range $b^{(1)}{*} < b < b_*^{(1)}$ there are three flows (due to $\gamma_3^{(1)}$ and $\lambda_3^{(1)}$ being two-valued) and, finally, when $b_*^{(1)} < b < 0$, there are two three-mode flows. Numerical-analytical investigations show that there are no other one-, two- or three-mode flows when $b_*^{(2)} < b < 0$. The pattern of possible flows which has been reported is unknown in the literature since its establishment using a speculative approach (from general considerations) is problematic and, in our opinion, hopeless.

The velocity profiles $y_k^{(1)}(x)$ for the first range of values of b have been constructed and studied in detail[19] for various values of the angle β. They correspond to the analysis presented above for $B = 10^\circ$. The functions $y_k^{(2)}(x)$ for the value $b = -60$ are shown in the upper part of Fig. 2; both of the four-mode profiles $y_4^{(2)\pm}(x)$ are represented. When the parameter b is increased, the spread of all the curves becomes considerably greater. It increases by an order of magnitude when $b = -10$ (the lower part of Fig. 2). Note that the total area of the figure under each curve is equal to unity (the normalized flow rate) with a very small error $z_k^{(2)}(1) \sim 10^{-5}$ (see (2.2)).

4. General characteristics of the flows of a viscous fluid in a diverging channel

The further investigation of a higher order bifurcation pattern involves the determination of the following triple points $b_*^{(n)}$, finding the ranges $b^{(n)*} \le b < b^{(n-1)*}$ and constructing the families of curves $\gamma_k^{(n)}$ and $\lambda_k^{(n)}$ as functions of the parameter b for $b^{(n)^*} \le b < 0$. We recall that the values of the index $n = 3, 4, \ldots$ indicate the number of the interval, while the possible values of the subscript k (the number of modes) is a function of the number of the range n. From the above analysis, it follows that $k = 2n - 1, 2n, 2n + 1$.

A fragment of the flow bifurcation pattern in the neighbourhood of $b = b_*^{(3)}$ for the third range $-176.5 \approx b^{(3)*} \le b < b^{(2)*}$ is shown in Fig. 3 which corresponds to with substantially greater values of the required quantities than in the preceding ranges. The behaviour of the family of curves $\gamma_k^{(3)}$ and $\lambda_k^{(2)}$, where $k = 5, 6, 7$ is similar to that studied above for $n = 1, 2$ when $k = 1, 2, 3$ and $k = 3, 4, 5$. Five- six- and seven-mode flows exist in the neighbourhood of the triple bifurcation point $b = b_*^{(3)} \approx -169.5$ and, moreover, $\gamma_5^{(3)} = 0$ when $b = b_*^{(3)}$. There are no flow patterns with a number of modes $k < 5$. The critical value $b = b^{(3)*}$ is the boundary of the range and the turning point of the curves $\gamma_7^{(3)}$ and $\lambda_7^{(3)}$. For $n = 3$, the curves $\gamma_k^{(n)}$ and $\lambda_k^{(n)}$ tend to the vertical asymptote when $b \to -0$ more rapidly than in the case when $n = 1$, 2.

There are qualitatively similar patterns for arbitrary $n \ge 4$. They are accompanied by an increase in the values of $\gamma_k^{(n)}$ and $\lambda_k^{(n)}$ and variations of the velocity profiles $y_k^{(n)}(x)$.

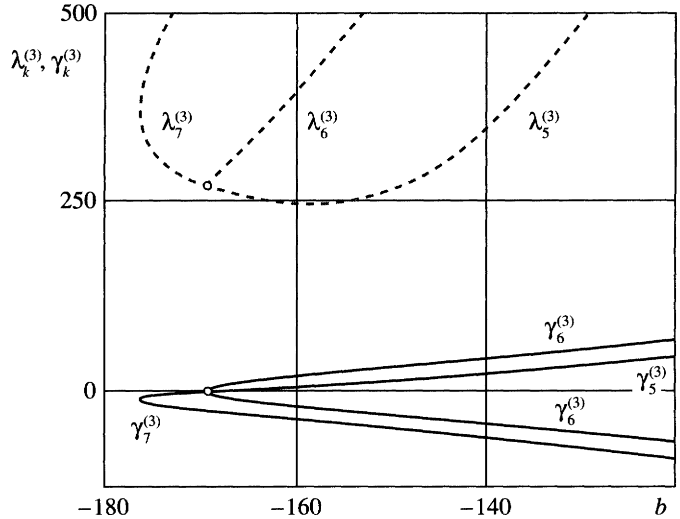

Fig. 3.

Analysis suggests that, in the case of an asymptotically small value $\beta(0 < \beta \lesssim 10^\circ)$, the critical points $b_*^{(n)}$, at which $\gamma_{2n-1}^{(n)} = 0$, are also the site of bifurcation of the flows (triple points) which are distributed according to the rule:

$$b_*^{(n)} = -6\pi n^2; \quad \mathrm{Re}_*^{(n)} = 3\pi n^2/\beta, \quad n^2/\beta \gg 1 \tag{4.1}$$

(see the cases $n = 1, 2, 3$ above). The dependence $b_*^{(1)}(b)$ is determined graphically in the whole domain of existence $0 < \beta \leq \pi/2$. The boundary points $b^{(n)*} < b_*^{(n)}$ are shifted by a small amount to the left (of the order of a few units), which increases appreciably as n becomes larger. Hence, the distance between the points $b_*^{(n+1)}$ and $b_*^{(n)}$ is approximately equal to $6\pi(2n+1)$., that is, it increases quite rapidly as n becomes larger. The ranges $b^{(n+1)*} - b^{(n)*}$ have the same length.

5. Conclusion and inferences

The solution of the Jeffery[1] problem has been constructed over a wide range of variation of the Reynolds number using the numerical-analytical method of accelerated convergence[18–20] for a specified aperture angle of the diverging channel. The bifurcations in the flow pattern, which are unknown in the literature, have been constructed and the non-uniqueness of the multimode flows has been established.

It has been shown that, when the Reynolds number is increased from the value $Re_*^{(1)} = |b_*^{(1)}|/(2\beta)$, there are no purely diverging single-mode flows. The single-mode flow, which is symmetrical about the channel axis, is replaced by a three-mode flow and there is a range of Re values for which two symmetric three-mode velocity profiles exist (see Fig. 1), which are generated by a bifurcation of the single-mode flow pattern. In the neighbourhood of the bifurcation point, the existence of two asymmetric two-mode flow patterns is established together with the three-mode flow patterns. The velocity profiles are antisymmetric to one another about the channel axis and exist for the values $0 < Re < Re_*^{(1)}$.

An increase in the Reynolds number leads to a second point of bifurcation $b_*^{(2)}$ (see relations (4.1) and Fig. 1) in the neighbourhood of which there is a single three-mode flow and two five-mode flows which are symmetrical about the channel axis as well as two four-mode flow patterns which are antisymmetric to one another. The pattern of change in the flow is similar to that considered for the first bifurcation value of the Reynolds number. Note that all the four-mode flow velocity profiles have a node on the channel axis; they exist when $Re < |b_*^{(2)}|/(2\beta) = Re_*^{(2)}$. There are two five-mode flow patterns in the range $|b_*^{(2)}| < 2\beta Re \leq |b^{(2)*}|$. When $\mathrm{Re}^{(2)*} = |b^{(2)*}|/(2\beta)$ and $\mathrm{Re} \leq \mathrm{Re}^{(2)*}$, there is a single five-mode and a single three-mode flows, which are generated by the second bifurcation point.

Calculations, corresponding to the third bifurcation point $b_*^{(3)}$ (4.1) have been presented (see Fig. 3). In the neighbourhood of this point, there are a five-mode, two six-mode and two seven-mode flow patterns. The flow patterns are similar to those indicated above. The odd-mode velocity profiles are symmetrical about the axis of the diverging channel. The even-mode (six-mode in the given case) processes turn out to be antisymmetric and they can be represented as a combination of even-mode flow-patterns of lower order (three two-mode flow-patterns in this case).

The basic properties of flows for the subsequent ranges can be described on the basis of a qualitative analysis of the flow of a viscous fluid for the first three ranges. We emphasize that, in the case of the first range of values $0 < Re < |b_*^{(1)}|$, in addition to the main single-mode purely diverging flow there are two pairs each of two-, three-, four- and so on multimode flow patterns. When the Reynolds number increase further, an analogous exclusion of flow-patterns with modes $k = 2n$, $2n - 1$ occurs, where n is the number of the bifurcation point and the interval.

Calculations suggest[18,19] that, when the semi-apertur angle is reduced $\beta \in (0, 10^\circ)$, the solutions of the Jeffery problem are found to be qualitatively and quantitatively very close. An increase in the angle $(10^\circ < \beta \leq 90^\circ)$ leads to solutions, the behaviour of which qualitatively corresponds to what has been described above. The main difference lies in the reduction in the values of $b_*^{(n)}$ and $b^{(n)*}$ and the corresponding values of the Reynolds number (see rule (4.1)). Considerable differences arise when $90^\circ < \beta \leq 180^\circ$ which corresponds to diverging channel aperture angles greater than for an unfolded diverging channel which requires additional investigation.

Acknowledgements

This research was financed by the Russian Foundation for Basic Research (06-01-00441, 08-01-00180, 08-01-00231, 08-08-0292) and the Programme for the Support of Leading Scientific Schools (NSh-4315.2008.1).

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Translated by E.L.S.